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High Density Effective Theory Confronts the Fermi Liquid

Simon Hands

Department of Physics, University of Wales Swansea,
Singleton Park, Swansea SA2 8PP, U.K.

Abstract

The high density effective theory recently introduced by Hong and Hsu to describe ultradense relativistic fermionic matter is used to calculate the tree-level forward scattering amplitude between two particles at the Fermi surface. While the direct term correctly reproduces that of the underlying gauge theory, the exchange term has the wrong sign. The physical consequences are discussed in the context of Landau's theoretical description of the Fermi liquid.

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1 Introduction

When studying any strongly-interacting system, an effective description which highlights the low energy degrees of freedom is highly desirable. Such a programme has recently been initiated by Hong and collaborators for the case of systems with a large number density of quarks, in which case the simplified description keeps as excitations about the ground state just those particle and hole states in the immediate vicinity of the Fermi surface. The resulting high density effective theory (HDET) [1] is derived from QCD in very much the same way as the heavy quark effective theory (HQET) [2], which has proved so successful in the description of bound states of heavy and light quarks. In HQET, heavy quarks of mass M moving with 4-velocity v_μ are described by fields h_v defined in terms of the original quark fields by

$$h_v(x) = e^{iMv^\mu x_\mu} P_v \psi(x) \quad \text{with} \quad P_v = \tfrac{1}{2}(1 + \not{v}\,). \tag{1.1}$$

The projector P_v enables the kinetic term for the h_v fields to be independent of the large scale M, while the phase factor ensures that even when the h_v fields are restricted to carry some residual momentum $p \ll M$, gluons which scatter off the heavy quark exchange momentum with a state carrying physical momentum $P \simeq O(M)$. HDET employs a similar manipulation with M replaced by the chemical potential μ, which for a degenerate system at zero temperature may be identified with the Fermi energy. It is then possible to devise a theory in which both particle and hole excitations appear with equal measure in the path integral [3].

The HDET approach has succeeded in calculating the screening mass of electric gluons in quark matter, and in calculating the strength of the gap induced by a BCS instability to diquark pairing in the color-flavor locked (CFL) channel [1]. Most intriguingly, in Euclidean metric it yields at leading order in Λ/μ (where Λ is the low energy scale of interest) a positive definite path integral measure [3]. This has enabled a Vafa-Witten style proof that the CFL phase is the true ground state of QCD at asymptotically high density [4], as well as raising the possibility that HDET might form a starting point for non-perturbative lattice simulations of dense matter.

There is, however, one important physical respect in which dense matter differs from the heavy-light systems described by HQET. In the latter case the only important interactions of the heavy quarks are small-angle scattering by t-channel exchange of gluons with light quarks. In quark matter scattering takes place between identical particles, implying that interactions in the u-channel, in which large momenta may be exchanged between two non-parallel particles at the Fermi surface, are also relevant.

In many-body and nuclear physics these two distinct contributions are known respectively as "direct" and "exchange" processes. While HQET and HDET both capture the essential physics of the direct interaction, for HDET one should also check its behaviour in the exchange channel.

After reviewing the version of HDET as set out in Refs. [1, 3] in the next section, I will outline in Sec. 3 an explicit calculation of the amplitude for forward scattering between two particle states at the Fermi surface, using both gauge theory and HDET, and show that while HDET successfully reproduces the direct term, there is a mismatch for the exchange. Forward scattering is interesting for degenerate systems, and indeed, is known to be the only non-irrelevant interaction at the Fermi surface in the renormalisation group sense [5]. In Sec. 4 I will show that this amplitude is a central feature in an alternative and much older phenomenological description of dense matter, the Fermi liquid [6]. The consequences of an incorrect description of exchange interactions for the detailed relations between Fermi liquid parameters such as the Fermi energy, momentum and velocity, as well as for the behaviour of collective excitations, are then set out in Sec. 5. Finally I present some brief conclusions.

2 High Density Effective Theory

We begin by reviewing the derivation of the high density effective theory (HDET), starting from the fermionic part of the gauge theory Lagrangian density and including a chemical potential μ (we neglect any bare quark mass):

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu + \mu\gamma_0)\psi(x) \tag{2.1}$$

where the covariant derivative $D_\mu \equiv \partial_\mu + igA_\mu$. Rewrite the physical quark 3-momentum as $\mathbf{k} = \mu\hat{\mathbf{p}} + \mathbf{p}$ where the "residual" momentum, giving the distance from the Fermi surface, satisfies $\mathbf{p} \parallel \mathbf{k}$. HDET assumes $|\mathbf{p}| \ll \mu$, and is expressed in terms of fields most naturally thought of in reciprocal space as functions of $\mathbf{p}$. This is achieved via the decomposition into "fast" ψ_- and "slow" ψ_+ degrees of freedom

$$\psi(x) = \exp(i\mu\mathbf{x}.\hat{\mathbf{p}})[\psi_+(x) + \psi_-(x)] \tag{2.2}$$

where $|\hat{\mathbf{p}}| = 1$ and $\psi_\pm$ are eigenstates of the projection operators

$$P_\pm(p) = \tfrac{1}{2}(1 \pm \vec{\alpha}.\hat{\mathbf{p}}). \tag{2.3}$$

The matrix $\vec{\alpha} = \gamma_0 \vec{\gamma}$, and where needed we use the representation

$$\vec{\alpha} = \begin{pmatrix} & \vec{\sigma} \\ \vec{\sigma} & \end{pmatrix}. \tag{2.4}$$

We also define a new covariant derivative $\tilde{D}_\mu$:

$$\tilde{D}_\mu = \partial_\mu + ige^{-i\mu\mathbf{x}.\hat{\mathbf{p}}} A_\mu e^{i\mu\mathbf{x}.\hat{\mathbf{p}}} \equiv \partial_\mu + ig\tilde{A}_\mu. \tag{2.5}$$

In real space the fields must be constructed using the operator $\hat{\mathbf{p}} = (-i/\sqrt{\nabla^2})\vec{\nabla}$. In terms of the new fields the quark Lagrangian (2.1) becomes

$$\begin{aligned} \mathcal{L} &= \bar{\psi}_+ \gamma_0 (1, \hat{\mathbf{p}})^\mu i\tilde{D}_\mu \psi_+ + \bar{\psi}_- \gamma_0 [(1, -\hat{\mathbf{p}})^\mu i\tilde{D}_\mu + 2\mu]\psi_- + \left[\bar{\psi}_- i\gamma_\perp^\mu \tilde{D}_\mu \psi_+ + h.c.\right] \\ &= \bar{\psi}_+ i\gamma_\parallel^\mu \tilde{D}_\mu \psi_+ + \bar{\psi}_- (i\gamma_\parallel^\mu \tilde{D}_\mu + 2\mu\gamma_0)\psi_- - g\left[\bar{\psi}_- \gamma_\perp^\mu \tilde{A}_\mu \psi_+ + h.c.\right] \end{aligned} \tag{2.6}$$

where we define $\gamma_\parallel^\mu = (\gamma_0, \hat{\mathbf{p}}\vec{\gamma}.\hat{\mathbf{p}})^\mu$, $\gamma_\perp^\mu = \gamma^\mu - \gamma_\parallel^\mu$, and the form of the final term follows from $\gamma_\parallel^\mu \partial_\mu \equiv \partial\!\!\!/$, $\gamma_\perp^\mu \partial_\mu = 0$. We have used the identities [1]

$$P_\mp \gamma^\mu P_\pm = \gamma_0 (1, \pm\hat{\mathbf{p}})^\mu P_\pm \quad ; \quad P_\pm \gamma^\mu P_\pm = \gamma_\perp^\mu P_\pm. \tag{2.7}$$

For $p \ll \mu$ it is possible to integrate out the fast modes ψ_- to yield a low energy effective Lagrangian written solely in terms of ψ_+. At tree level this is done via the equation of motion

$$\psi_- = \frac{g\gamma_0}{2\mu + i\overline{D}_\parallel} \gamma_\perp^\mu \tilde{A}_\mu \psi_+ = \frac{g\gamma_0}{2\mu} \sum_{n=0}^{\infty} \left(-\frac{i\overline{D}_\parallel}{2\mu} \right)^n \gamma_\perp^\mu \tilde{A}_\mu \psi_+ \tag{2.8}$$

where $\overline{D}_\parallel = (1, -\hat{\mathbf{p}})^\mu \tilde{D}_\mu$. We obtain the HDET effective Lagrangian as a derivative expansion, ie. in powers of Λ/μ where $\Lambda \ll \mu$ denotes a momentum scale of physical interest:

$$\mathcal{L}_{HDET} = \bar{\psi}_+ i\gamma_\parallel^\mu \tilde{D}_\mu \psi_+ + \frac{g^2}{2\mu} \bar{\psi}_+ \gamma_0 (\gamma_\perp^\mu \tilde{A}_\mu)^2 \psi_+ + O\left(\frac{D^2}{\mu^2}\right). \tag{2.9}$$

The leading order term of HDET should therefore be able to describe the dynamics of low energy excitations near the Fermi surface, in which any momentum transferred by scattering off a gluon should be much smaller than the Fermi momentum. At higher order, further terms in the HDET action such as a four-fermi contact interaction and a gluon screening mass are generated [1].

In fact, HDET has to be subtly modified in order to preserve unitarity. The ψ_- excitations are nothing other than anti-particles, whose physical impact in superdense

matter with $p \ll \mu$ is relatively unimportant. However, just as in a metal or a semiconductor, we must also take hole excitations in the Fermi sea into account, which have identical quantum numbers to anti-particles, but play a much more important role. In order to do this consistently we modify the definitions (2.2, 2.5) to read [3]:

$$\psi(x) = e^{iX}\psi_+(x) \quad ; \quad \tilde{A}_\mu(x) = e^{-iX}A_\mu(x)e^{iX} \quad \text{with} \quad X \equiv \mu(\mathbf{x}.\hat{\mathbf{p}})(\vec{\alpha}.\hat{\mathbf{p}}). \tag{2.10}$$

Note that $\psi_+(p)$ can now satisfy either of $P_\pm(p)\psi_+ = \psi_+$ depending on whether the physical state it describes is a particle or a hole, in either case with momentum $\mathbf{p}$. It is not possible, however, to derive HDET from the original gauge theory starting with the projection (2.10). The quark-gluon vertex arising from the leading order term $\bar{\psi}_+ i\gamma_\parallel^\mu \tilde{D}_\mu \psi_+$ in $\mathcal{L}_{HDET}$ now reads

$$\mathcal{L}_{qqg} = -g\int_x A^\mu(q)e^{iqx}\bar{\psi}_+(p')e^{-ip'x}e^{-i\mu\mathbf{x}.\hat{\mathbf{p}}'\vec{\alpha}.\hat{\mathbf{p}}'}(\gamma_0, \hat{\mathbf{p}}.\vec{\gamma}\hat{\mathbf{p}})_\mu e^{ipx}e^{i\mu\mathbf{x}.\hat{\mathbf{p}}\vec{\alpha}.\hat{\mathbf{p}}}\psi_+(p). \tag{2.11}$$

After integrating over x we find

$$\begin{aligned} \mathcal{L}_{qqg} &= -gA^\mu(q)\Big[\bar{\psi}_+(p')P_-(p')\gamma_0(1,+\hat{\mathbf{p}})_\mu P_+(p)\psi_+(p)\delta^4\Big((p'+\mu\hat{\mathbf{p}}') - (p+\mu\hat{\mathbf{p}}) - q\Big) \\ &+ \bar{\psi}_+(p')P_+(p')\gamma_0(1,-\hat{\mathbf{p}})_\mu P_-(p)\psi_+(p)\delta^4\Big((p'-\mu\hat{\mathbf{p}}') - (p-\mu\hat{\mathbf{p}}) - q\Big) \\ &+ \bar{\psi}_+(p')P_-(p')\gamma_0(1,-\hat{\mathbf{p}})_\mu P_-(p)\psi_+(p)\delta^4\Big((p'+\mu\hat{\mathbf{p}}') - (p-\mu\hat{\mathbf{p}}) - q\Big) \\ &+ \bar{\psi}_+(p')P_+(p')\gamma_0(1,+\hat{\mathbf{p}})_\mu P_+(p)\psi_+(p)\delta^4\Big((p'-\mu\hat{\mathbf{p}}') - (p+\mu\hat{\mathbf{p}}) - q\Big)\Big], \end{aligned} \tag{2.12}$$

the four terms describing respectively particle or a hole scattering off a gluon, and particle-hole creation and annihilation. The effect of the phase factor in (2.10) has been to ensure that the gluon scatters off states carrying physical rather than residual momenta. The Dirac structure in principle is given by expressions such as

$$P_-(p')\gamma_0 P_+(p) = \frac{\gamma_0}{4}\left[1 + \hat{\mathbf{p}}'.\hat{\mathbf{p}} + \vec{\alpha}.(\hat{\mathbf{p}}' + \hat{\mathbf{p}}) + i\gamma_5\vec{\alpha}.\hat{\mathbf{p}}' \times \hat{\mathbf{p}}\right] \tag{2.13}$$

Note, however, that since by assumption $\hat{\mathbf{p}}'.\hat{\mathbf{p}} = 1 + O(p^2/\mu^2)$, $\hat{\mathbf{p}}' \times \hat{\mathbf{p}} = O(p/\mu)$, at leading order in HDET we can consistently approximate

$$P_\mp(p')\gamma_0 P_\pm(p) \approx \gamma_0 P_\pm(p). \tag{2.14}$$

In this same limit $P_\pm(p')\gamma_0 P_\pm(p) \approx 0$.

In Euclidean metric, the leading term of the HDET Lagrangian (2.9) is of the form $\bar{\psi}_+ M\psi_+$ where M is both anti-hermitian and anticommutes with γ_5, implying

that the functional measure of the path integral $\det M$ is real and positive. This raises the attractive possibility of using the HDET action as a basis for non-perturbative simulations of systems at asymptotically high quark densities using lattice Monte Carlo methods [3, 4]. The higher order terms in (2.9) describing corrections of $O(\Lambda/\mu)$ are of the form $\bar{\psi}_+ H \psi_+$ where H is hermitian. In general, $\det(M+H)$ is complex, leading to a restoration of the notorious "sign problem" at finite density. It may prove possible, however, to find means to treat corrections for $\Lambda/\mu \lesssim 1$, much as corrections to QCD simulations at $\mu = 0$ can be calculated for $\mu/T \lesssim 1$ [7].

3 Forward Scattering

Let us consider a very simple process, tree-level forward scattering of two relativistic particles at the Fermi surface, by comparing calculations made using both HDET and the full underlying quantum field theory, which for simplicity we will initially assume to be QED. The relevant Feynman diagrams are shown in Fig. 1. In Minkowski metric

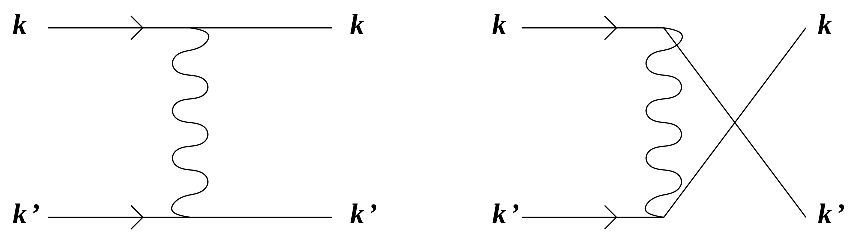

Figure 1: Feynman diagrams for forward scattering, showing direct (left) and exchange (right) contributions.

and Feynman gauge the QED scattering amplitude for the direct process is

$$\mathcal{A}^{dir}(k,k') = \frac{1}{4\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}} \mathcal{M}^{dir}(\mathbf{k},\mathbf{k}') \tag{3.1}$$

with

$$i\mathcal{M}^{dir} = ie^2 \frac{g^{\mu\nu}}{-\lambda^2} \bar{u}(k,s)\gamma_\mu u(k,s)\bar{u}(k',s')\gamma_\nu u(k',s'). \tag{3.2}$$

Here, e is the electron charge, an infra-red regulator mass λ has been introduced for the photon, and u, $\bar{u}$ are plane-wave spinors normalised to $\bar{u}u = 2m$, with m the electron mass. The four-momentum $k_\mu = (\varepsilon_{\mathbf{k}}, \mathbf{k})_\mu$, with the energy $\varepsilon_{\mathbf{k}}$ of course equal to μ at the Fermi surface.

To evaluate the spin-symmetric part of the amplitude we average over s, s' to obtain

$$\mathcal{A}^{dir}(k,k') = -\frac{1}{16\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}}\frac{e^2}{\lambda^2}\mathrm{tr}[\gamma_\mu(\not{k}+m)]\mathrm{tr}[\gamma^\mu(\not{k}'+m)] = -\frac{e^2}{\lambda^2}\frac{\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}-\mathbf{k}.\mathbf{k}'}{\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}}. \quad (3.3)$$

The exchange contribution includes a relative minus sign due to Fermi statistics:

$$\begin{aligned}\mathcal{A}^{ex}(k,k') &= -\frac{1}{16\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}}\frac{e^2}{(k-k')^2-\lambda^2}\mathrm{tr}[\gamma_\mu(\not{k}'+m)\gamma^\mu(\not{k}+m)] \\ &= \frac{1}{2\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}}\frac{e^2}{(\mathbf{k}-\mathbf{k}')^2+\lambda^2}[-(\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}'}-\mathbf{k}.\mathbf{k}')+2m^2]. \end{aligned} \quad (3.4)$$

Note that $k_0 = k_0'$ for two states at the Fermi surface. In the limit $m, \lambda \to 0$ we have $\varepsilon_{\mathbf{k}} = |\mathbf{k}| = \mu$ and [8]

$$\mathcal{A}^{dir}_{QED}(k,k') = -\frac{e^2}{\lambda^2}(1-\cos\theta), \quad (3.5)$$

$$\mathcal{A}^{ex}_{QED}(k,k') = -\frac{e^2}{4\mu^2}, \quad (3.6)$$

where θ is the angle between the two particle momenta.

To repeat the calculation using HDET a necessary ingredient are the plane wave states. Eigenstates $\Psi_\pm$ of the momentum space HDET Dirac equation

$$p_0\Psi_\pm(p) = \vec{\alpha}.\mathbf{p}P_\pm(p)\Psi_\pm(p) \quad (3.7)$$

are given by

$$\begin{aligned}\Psi_+(p,s) &= \begin{pmatrix} \phi^s \\ +\vec{\sigma}.\hat{\mathbf{p}}\,\phi^s \end{pmatrix} \text{ with eigenvalue } p_0 = +|\mathbf{p}|; \\ \Psi_-(p,s) &= \begin{pmatrix} \chi^s \\ -\vec{\sigma}.\hat{\mathbf{p}}\,\chi^s \end{pmatrix} \text{ with eigenvalue } p_0 = -|\mathbf{p}|. \end{aligned} \quad (3.8)$$

The two-spinors ϕ, χ are given by $\phi^1 = \chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\phi^2 = \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, whence

$$\Psi^\dagger_\pm\Psi_\pm = 2 \quad ; \quad \bar{\Psi}_\pm\Psi_\pm = 0 \quad ; \quad \sum_s \Psi_\pm(p,s)\bar{\Psi}_\pm(p,s) = \gamma_\mu(1,\pm\hat{\mathbf{p}})^\mu. \quad (3.9)$$

Note also the important identity

$$\Psi_\pm(p,s) \equiv \Psi_\mp(-p,-s). \quad (3.10)$$

Now let's use HDET, and in particular the qqA vertex described by the first term of (2.12), to calculate the spin-symmetric forward scattering amplitude between two particles at the Fermi surface. First we examine the direct term; with the normalisation (3.9) we have

$$\begin{aligned}
\mathcal{A}^{dir}_{HDET}(p,p') &= \frac{1}{16}\sum_{s,s'}\mathcal{M}^{dir}(p,p') \\
&= \frac{e^2}{16}\frac{g^{\mu\nu}}{-\lambda^2}\sum_{s,s'}\bar{\Psi}_+(p,s)\gamma_0(1,+\hat{\mathbf{p}})_\mu\Psi_+(p,s)\bar{\Psi}_+(p',s')\gamma_0(1,+\hat{\mathbf{p}}')_\nu\Psi_+(p',s') \\
&= -\frac{e^2}{16\lambda^2}\mathrm{tr}[\gamma_0\gamma_\rho]\mathrm{tr}[\gamma_0\gamma_\sigma](1,\hat{\mathbf{p}})^\rho(1,\hat{\mathbf{p}}')^\sigma(1-\hat{\mathbf{p}}.\hat{\mathbf{p}}') = \mathcal{A}^{dir}_{QED}(p,p'). \quad (3.11)
\end{aligned}$$

The direct term calculated in HDET thus agrees with that of the full theory (3.5). For the exchange term,

$$\mathcal{A}^{ex}_{HDET}(p,p') = -\frac{1}{16}\sum_{s,s'}\frac{e^2}{q^2-\lambda^2}\bar{\Psi}_+(p',s')\gamma_0(1,\hat{\mathbf{p}})^\mu\Psi_+(p,s)\bar{\Psi}_+(p,s)\gamma_0(1,\hat{\mathbf{p}}')_\mu\Psi_+(p',s'), \tag{3.12}$$

with $\mathbf{q} = (|\mathbf{p}'| + \mu)\hat{\mathbf{p}}' - (|\mathbf{p}| + \mu)\hat{\mathbf{p}}$ (as before, for two states near the Fermi surface $q_0 \approx 0$). Performing the average over spins and resulting trace we find

$$\mathcal{A}^{ex}_{HDET}(p,p') = -\frac{e^2}{4q^2}[2g_{\rho 0}g_{\sigma 0} - g_{00}g_{\rho\sigma}](1,\hat{\mathbf{p}})^\rho(1,\hat{\mathbf{p}}')^\sigma(1-\hat{\mathbf{p}}.\hat{\mathbf{p}}') = -\frac{e^2}{4q^2}(1-\cos^2\theta). \tag{3.13}$$

Finally, $q^2 = -2\mu^2(1-\cos\theta)(1+O(p/\mu))$, so that to leading order the result is

$$\mathcal{A}^{ex}_{HDET}(p,p') = \frac{e^2}{8\mu^2}(1+\cos\theta). \tag{3.14}$$

If the calculation is repeated with the more accurate vertex factor (2.13), then the answer is

$$\mathcal{A}^{ex}_{HDET}(p,p') = \frac{e^2}{8\mu^2}\left[\frac{(1+\cos\theta)^2}{2} + \frac{1}{8}\sin^2\theta(1-\cos\theta)\right]. \tag{3.15}$$

As anticipated, this result agrees with (3.14) up to terms of $O(p^2/\mu^2)$. In either case, though, the HDET result, though of the same order of magnitude, actually disagrees even in overall sign with that of QED (3.6).

Since QED and HDET disagree even at leading order in p/μ, it is difficult to see how the discrepancy between (3.6) and (3.14) can be sorted out at higher order; indeed, the next term $(2\mu)^{-1}\bar{\psi}_+\gamma_0(e\tilde{\not{A}}_\perp)^2\psi_+$ in the derivative expansion would correct $\mathcal{A}^{ex}_{HDET}$ only at $O(e^4)$. In fact, to reconcile the two approaches we need to step back

to the mixed term in (2.6) before ψ_- is eliminated: $e[\bar{\psi}_-\tilde{\not{A}}_\perp\psi_+ + h.c.]$. This yields a vertex $-ie\gamma_0(0,\vec{\alpha}-\hat{\mathbf{p}})_\mu$, which can contribute to particle – particle scattering once we realise that an anti-particle created by $\bar{\psi}_-(p)$ is indistinguishable from a particle with momentum $-p$. The new vertex does not contribute to direct scattering, but does yield three additional contributions to $\mathcal{A}^{ex}$ at $O(e^2)$, eg:

$$\begin{aligned}
\mathcal{A}^{ex}_{+-} &= -\frac{e^2}{16q^2}\sum_{s,s'}\bar{\Psi}_+(p',s')\gamma_0(1,\hat{\mathbf{p}})^\mu\Psi_+(p,s)\bar{\Psi}_-(-p,-s)\gamma_0(0,\vec{\alpha}-\hat{\mathbf{p}}')_\mu\Psi_+(p',s') \\
&= -\frac{e^2}{4q^2}\Big[-p'_i(g_{\rho0}g_{\sigma i}+g_{\rho i}g_{\sigma0})+\hat{\mathbf{p}}.\hat{\mathbf{p}}'(2g_{\rho0}g_{\sigma0}-g_{\rho\sigma}g_{00})\Big](1,\hat{\mathbf{p}}')^\rho(1,\hat{\mathbf{p}})^\sigma \\
&= \frac{e^2}{8\mu^2}\frac{(1+\cos\theta)^2}{1-\cos\theta},
\end{aligned} \tag{3.16}$$

where use has been made of (3.10). We then find that to $O(e^2)$

$$\mathcal{A}^{ex}_{++}+\mathcal{A}^{ex}_{+-}+\mathcal{A}^{ex}_{-+}+\mathcal{A}^{ex}_{--}=\mathcal{A}^{ex}_{QED} \tag{3.17}$$

where ++ denotes the original HDET term (3.14). We conclude that once the ψ_- degrees of freedom have been eliminated to define the effective theory solely in terms of ψ_+, the exchange amplitude (3.6) cannot be faithfully reproduced at tree level.

4 The Fermi Liquid

Why is this important? The forward scattering amplitude plays a central role in another, much older phenomenological description of degenerate matter, the Fermi liquid [6, 9]. The essential physical idea is that the dominant low-energy excitations in the neighbourhood of the Fermi surface are quasiparticle states having energy $\varepsilon_{\mathbf{k}}$, width of $O(\varepsilon_{\mathbf{k}}-\mu)^2$ and equilibrium distribution

$$n_{\mathbf{k}} = \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}}-\mu}{T}\right)+1}. \tag{4.1}$$

For temperature $T=0$ we expect $\varepsilon_{\mathbf{k}}$ to have the form

$$\varepsilon_{\mathbf{k}} \simeq \mu + \beta_F(|\mathbf{k}|-k_F) \tag{4.2}$$

where k_F, $\beta_F \equiv |\vec{\nabla}_{\mathbf{k}}\varepsilon_{|\mathbf{k}|=k_F}|$ are respectively the Fermi momentum and Fermi velocity, which at this level are phenomenological parameters. Leading order HDET assigns them the free-field values $k_F=\mu$, $\beta_F=1$. Fermi liquid theory derives quantitative

power via the equation for the variation in quasiparticle energy in response to a departure δn from equilibrium:

$$\delta\varepsilon_{\mathbf{k}} = \int \frac{d^3k'}{(2\pi)^3} \mathcal{F}_{\mathbf{k},\mathbf{k}'} \delta n_{\mathbf{k}'}. \tag{4.3}$$

The Fermi liquid interaction $\mathcal{F}_{\mathbf{k},\mathbf{k}'}$ is given to lowest order in perturbation theory by [10, 11]

$$\mathcal{F}_{\mathbf{k},\mathbf{k}'} = -\mathcal{A}(\mathbf{k}, \mathbf{k}') = -\mathcal{A}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \tag{4.4}$$

where in the last step we have assumed that the states are so close to the Fermi surface that the matrix element only depends on the relative orientation of the particle momenta. If $\mathcal{F}$ can be calculated in some scheme, it is then possible to obtain quantitative relations between the parameters μ, k_F and β_F, as well as derive the velocities of collective excitations [6, 8, 9].

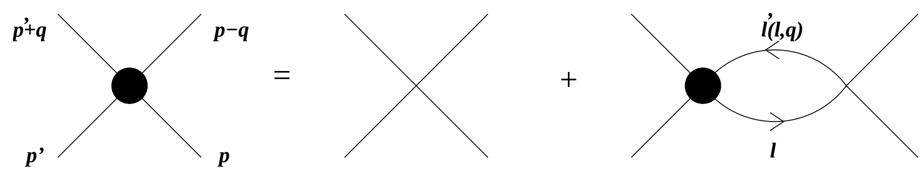

Figure 2: Schwinger-Dyson equation for the scattering amplitude $-\Gamma(p', p, q)$.

It may be helpful to outline the origin of the relation (4.4) – to my knowledge there is no argument more elegant than the original one presented by Landau in one of the earliest papers on non-perturbative field theory [10, 11]. Consider quasiparticle scattering with the kinematics sketched in Fig. 2: the full amplitude is then given by

$$\Gamma(p', p, q) = \Gamma^0(p', p) - i \int \frac{d^4\ell}{(2\pi)^4} \Gamma(p', \ell, q) S(\ell) S(\ell'(\ell, q)) \Gamma^0(q, p). \tag{4.5}$$

We wish to focus on the singular behaviour of Γ as the momentum transfer $q \to 0$, so separate out the part Γ^0 (shown as a simple point vertex in Fig. 2) which is regular in this limit. The singular behaviour results from the quasiparticle propagator product $S(\ell)S(\ell')$, and in particular from the contribution of a particle-hole pair, in which no net quark number flows through the diagram, for loop momenta on the Fermi surface. Using HDET propagators which are functions of residual rather than

physical momenta, but with spinor indices suppressed for simplicity, this product contains terms like

$$S(\ell)S(\ell'(\ell,q)) = \mathfrak{g} \times \frac{1}{\ell_0 - |\mathbf{l}| - i\epsilon} \; \frac{1}{(\omega - \ell_0) + (|\mathbf{l}| - |\mathbf{q}|\cos\theta) + i\epsilon} \tag{4.6}$$

where we have written $\omega = -q_0$, the $i\epsilon$ terms are chosen so that the first factor descibes a particle of momentum ℓ, the second a hole of momentum $-\ell'$, and the kinematics enforced by eg. the δ-functions of the third or fourth terms of (2.12) are such that $|\mathbf{l}'| = -|\mathbf{l}| + |\mathbf{q}|\cos\theta$, where θ is the angle between $\mathbf{q}$ and $\mathbf{l}$ and we have assumed all momenta $\ll \mu$ so that the curvature of the Fermi surface can be neglected and $\mathbf{l}' \parallel \mathbf{l}$. The factor $\mathfrak{g}$ arises from a trace over internal degrees of freedom, and counts the degeneracy of each momentum state. Within the integral in (4.5) the product may be approximated by

$$S(\ell)S(\ell') \simeq 2\pi\mathfrak{g}i\frac{|\mathbf{q}|\cos\theta}{\omega - |\mathbf{q}|\cos\theta}\delta(\ell_0)\delta(|\mathbf{l}|) + G(\ell) \tag{4.7}$$

where the pole structure has resulted in the first term being supported only for loop momenta corresponding to physical momenta on the Fermi surface, and the second term G is once again regular as $q \to 0$. The factor in the numerator arises because the kinematics restricts the integral over 3-momentum to $0 \leq |\mathbf{l}| \leq |\mathbf{q}|\cos\theta$.

The rest of the argument proceeds by examining the resulting integral equation

$$\begin{aligned}\Gamma(p',p,q) = \Gamma^0(p',p) \quad &- \quad i\int \frac{d^4\ell}{(2\pi)^4}\Gamma(p',\ell,q)G(\ell)\Gamma^0(\ell,p) \\ &+ \quad \frac{\mathfrak{g}k_F^2}{(2\pi)^3}\int d\Omega\Gamma(p',\ell,q)\Gamma^0(\ell,p)\frac{|\mathbf{q}|\cos\theta}{\omega - |\mathbf{q}|\cos\theta}\end{aligned} \tag{4.8}$$

in two distinct kinematic regimes as $q \to 0$. First consider $|\mathbf{q}|/\omega \to 0$:

$$\Gamma^\omega(p',p) \equiv \lim_{q\to 0,\, |\mathbf{q}|/\omega \to 0}\Gamma(p',p,q) = \Gamma^0(p',p) - i\int\frac{d^4\ell}{(2\pi)^4}\Gamma^\omega(p',\ell)G(\ell)\Gamma^0(\ell,p). \tag{4.9}$$

In fact, it is possible to eliminate dependence on the regular functions Γ^0 and G by substituting (4.9) in (4.8) to yield

$$\Gamma(p',p,q) = \Gamma^\omega(p',p) + \frac{\mathfrak{g}k_F^2}{(2\pi)^3}\int d\Omega\Gamma(p',\ell,q)\Gamma^\omega(\ell,p)\frac{|\mathbf{q}|\cos\theta}{\omega - |\mathbf{q}|\cos\theta}. \tag{4.10}$$

Now we can consider the opposite limit $\omega/|\mathbf{q}| \to 0$ in (4.10):

$$\Gamma^k(p',p) \equiv \lim_{q\to 0,\, \omega/|\mathbf{q}|\to 0}\Gamma(p',p,q) = \Gamma^\omega(p',p) - \frac{\mathfrak{g}k_F^2}{(2\pi)^3}\int d\Omega\Gamma^k(p',\ell)\Gamma^\omega(\ell,p). \tag{4.11}$$

The function Γ^k describes forward scattering for quasiparticles located on the Fermi surface, for which all physical scattering processes have $\omega = 0$, and can be identified with $-\mathcal{A}$ calculated in the previous section. In order to find a physical interpretation of Γ^ω, we consider (4.10) in the vicinity of a pole of $\Gamma(p', p, q)$ considered as a function of q; in this case the first term on the RHS can be neglected and the argument p' which plays a passive role can be suppressed. At the Fermi surface the residual momenta p and ℓ can be replaced by their corresponding unit 3-vectors. With the definition

$$\Phi(\hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}.\mathbf{q}}{\omega - \hat{\mathbf{p}}.\mathbf{q}} \Gamma(p, q) \tag{4.12}$$

eqn. (4.10) becomes an integral equation for excitations propagating with wavevector $\mathbf{q}$ and amplitude Φ:

$$(\omega - \hat{\mathbf{p}}.\mathbf{q})\Phi(\hat{\mathbf{p}}) = (\hat{\mathbf{p}}.\mathbf{q}) \frac{\mathfrak{g} k_F^2}{(2\pi)^3} \int d\Omega \Gamma^\omega(\hat{\mathbf{l}}, \hat{\mathbf{p}}) \Phi(\hat{\mathbf{l}}). \tag{4.13}$$

This equation[1] describes collective excitations of the shape of the Fermi surface known as zero sound, one of the principal predictions of the Fermi liquid theory [12], with Γ^ω playing the role of Fermi liquid interaction $\mathcal{F}$. Eqn. (4.11) thus yields the full relation between $\mathcal{A}$ and $\mathcal{F}$: to lowest order in perturbation theory we recover (4.4).

5 HDET Confronts the Fermi Liquid

There are many situations in strong interaction models, such as the chiral limit of the NJL model [13], or the color symmetric channel in QCD [8], where the direct contribution to $\mathcal{A}$ and hence $\mathcal{F}$ vanishes. In this situation HDET would yield a Fermi liquid interaction of the wrong sign, with potentially unphysical consequences.

To see how this works, define the spin-symmetric Landau parameters

$$f_l^S \equiv (2l + 1) \int \frac{d\Omega}{4\pi} P_l(\cos\theta) \mathcal{F}_{\mathbf{k},\mathbf{k}'}(\theta) \tag{5.1}$$

where the integral ranges over all angles between $\mathbf{k}$ and $\mathbf{k}'$ taken on the Fermi surface. I now quote without proof the following relations for a relativistic Fermi liquid [8]:

$$\beta_F = \frac{k_F}{\mu} - \frac{\mathfrak{g} k_F^2}{6\pi^2} f_1^S; \tag{5.2}$$

$$\frac{\partial \mu}{\partial n} = \frac{2\pi^2}{\mu \mathfrak{g} k_F} + f_0^S - \tfrac{1}{3} f_1^S. \tag{5.3}$$

[1] If proper account is taken of the absence of full Lorentz invariance, then all 3-momenta in the argument leading to (4.13) are rescaled by a factor β_F.

Here $n = \mathfrak{g}k_F^3/6\pi^2$ is the quark number density. Generalising the exchange result (3.6) to QCD with N_c colors and N_f flavors of massless quark (note the direct term vanishes in the color symmetric channel) we have

$$f_0^S = g^2\frac{(N_c^2-1)}{8N_c^2N_f\mu^2} \;\;;\;\; f_1^S = 0 \;\;;\;\; \mathfrak{g} = 2N_cN_f. \tag{5.4}$$

Integrating (5.3) we thus find to $O(g^2)$:

$$\beta_F = \frac{k_F}{\mu} = 1 - g^2\frac{(N_c^2-1)}{24N_c\pi^2}. \tag{5.5}$$

For the HDET exchange interaction (3.14), however, the corresponding results are

$$f_0^S = f_1^S = -g^2\frac{(N_c^2-1)}{16N_c^2N_f\mu^2} \tag{5.6}$$

implying

$$\beta_F = \frac{k_F}{\mu}\left(1 + g^2\frac{(N_c^2-1)}{48N_c\pi^2}\right) = 1 + g^2\frac{5(N_c^2-1)}{144N_c\pi^2}. \tag{5.7}$$

Thus the Fermi velocity is predicted to be superluminal in this approach.

At $O(g^2)$ we can correct for the mismatch between (5.5) and (5.7) by modifying the leading order HDET Lagrangian to

$$\mathcal{L}'_{HDET} = \bar{\psi}_+(\gamma_0, c_1\hat{\mathbf{p}}\vec{\gamma}.\hat{\mathbf{p}})^\mu i\tilde{D}_\mu\psi_+ \quad \text{with} \quad \tilde{A}_\mu = e^{-iX'}A_\mu e^{iX'} \quad ; \quad X' = c_2X, \tag{5.8}$$

with the constants chosen to be

$$c_1 = 1 - g^2\frac{11(N_c^2-1)}{144N_c\pi^2} \quad ; \quad c_2 = 1 - g^2\frac{(N_c^2-1)}{18N_c\pi^2}. \tag{5.9}$$

However, whilst it is feasible to repair HDET to reproduce parameters directly related to the fundamental quanta, there are other phenomena where the failure to deal with the exchange interaction correctly may be more serious. For instance, in theories where the only non-trivial Landau parameter is f_0^S, eqn. (4.13) may be solved for the zero sound velocity $\beta_0 = \omega/|\mathbf{q}|$ [12]:

$$\beta_0 = \beta_F\left(1 + 2\exp\left(-\frac{4\pi^2\beta_F}{\mathfrak{g}k_F^2 f_0^S}\right)\right). \tag{5.10}$$

The collective excitations are thus extremely sensitive to the sign of f_0^S.

6 Conclusion

The main result of this paper is that if the ψ_- degrees of freedom defined by the projection (2.2) are eliminated, the resulting effective theory incorrectly describes exchange interactions between quasiparticle states at the Fermi surface at tree level, as exemplified by the explicit perturbative QED calculation of Sec. 3.

It should be stressed that this result does not invalidate the HDET programme; the failure to reproduce exchange amplitudes may be compensated by introducing a four-fermi contact term of $O(g^2/\mu^2)$, as outlined by Schäfer [14]. In his approach all operators consistent with the symmetries of the underlying gauge theory are included from the start, and coefficients systematically determined by matching at a suitably chosen scale $\Lambda \sim \mu$. Within the approach adopted here in which the leading order terms are "derived" via the projection (2.10), the correction arises from integrating out ψ_- states from the loop of Fig. 2, but with the naive power counting of $O(g^4/\mu^2)$ non-perturbatively enhanced to $O(g^2/\mu^2)$.

How serious a problem is this? It is worth recalling that currently few people believe that quark matter is a Fermi liquid; rather it is supposed to be a superconductor with energy gap $\Delta \sim O(10)$MeV at the Fermi surface. The infra-red singularity leading to the key relation (4.11) for the Fermi liquid interaction is thus cut off, and the dire consequences predicted in Sec. 5 no longer hold. The relativistic degenerate electrons in the interior of a white dwarf are expected to form a Fermi liquid, but in this case a photon screening mass $\lambda \sim O(e\mu)$ should be included in the calculation, which implies that the exchange amplitude (3.6) is only $O(\alpha)$ compared to the direct term.

At the very least though, these considerations suggest that careful physical arguments should be supplied before the HDET approach is applied, particularly since one of the main goals of any conceivable numerical implementation would be to examine the parameter range of its applicability.

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